Stochastic resonance: A chaotic dynamics approach

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For a class of multistable systems it follows from qualitative results of Melnikov theory that deterministic and stochastic excitations play equivalent roles in the promotion of chaos. We use such results to suggest: (1) a method for assessing the role of the noise spectrum in enhancing the signal-to-noise ratio (SNR), the most effective spectral shape being that for which the power is distributed closest to the frequency of the Melnikov scale factor's peak; (2) a method for more effective SNR enhancement than can be achieved by increasing the noise, wherein the noise is left unchanged and a harmonic excitation with frequency based on the system's Melnikov scale factor is added to the system. The effectiveness of our Melnikov-based methods is confirmed by numerical simulations. The principle of a practical and effective nonlinear transduction device for enhancing SNR is proposed and demonstrated numerically. [S1063-651X(96)10808-4]

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I. INTRODUCTION

For a class of multistable systems with noise and a periodic signal, the improvement of the signal-to-noise ratio (SNR) [1] achieved by increasing the noise intensity is known as stochastic resonance (SR) [2,3] (or, as it will be referred to in this paper, classical SR). The essence of the physical mechanism underlying classical SR can be described as follows [4]. Consider the motion in a bistable double-well potential of a lightly damped particle subjected to stochastic excitation and a harmonic excitation (i.e., a signal) with low frequency ω_0. The signal is assumed to have small enough amplitude that, by itself (i.e., in the absence of the stochastic excitation), it is unable to move the particle from one well to another. We denote the characteristic rate, that is, the escape rate from a well under the combined effects of the periodic excitation and the noise, by $\alpha=2\pi n_{\mathrm{tot}}/T_{\mathrm{tot}}$, where n_{tot} is the total number of exits from a well during time T_{tot}. We consider the behavior of the system as we increase the noise while the signal amplitude and frequency are unchanged. For zero noise, $\alpha=0$, as noted earlier. For very small noise we have $\alpha<\omega_0$. As the noise increases, the ordinate of the spectral density of the output noise at the frequency ω_0, denoted by $\Phi_n(\omega_0)$, and the characteristic rate α increase. Experimental and analytical studies show that, until $\alpha\approx\omega_0$, a cooperative effect (i.e., a synchronizationlike phenomenon, as referred to in [5]) occurs wherein the signal output power $\Phi_s(\omega_0)$ increases as the noise intensity increases. Remarkably, the increase of $\Phi_s(\omega_0)$ with noise is faster than that of $\Phi_n(\omega_0)$. This results in an enhancement of the SNR. The synchronizationlike phenomenon plays a key role in the mechanism just described.

In this paper we offer an interpretation of stochastic resonance from a chaotic dynamics viewpoint. This viewpoint allows the use of Melnikov theory, which is applicable to a wide class of multistable systems whose Hamiltonian counterparts have homoclinic or heteroclinic orbits. Melnikov theory provides a necessary condition for the occurrence of chaos featuring irregular escapes from the wells. The theory was originally developed for deterministic systems with harmonic excitation [6]. It was subsequently extended to those systems' quasiperiodically excited counterparts [7]. On the basis of that extension it has been shown that Melnikov theory is also applicable to systems with stochastic forcing [8].

Melnikov theory yields qualitative results on the basis of which useful inferences can be made on system behavior even in the absence of a comprehensive mathematical apparatus such as has been developed for certain aspects of classical SR. In this paper we use the following consequence of Melnikov theory: for a wide class of systems, deterministic and stochastic excitations play qualitatively equivalent roles in inducing chaotic motions with escapes over a potential barrier, the motions being in both cases topologically conjugate to a shift map. Such motions therefore possess common qualitative features that suggest the extension of SR approaches beyond classical SR, so that the SNR can alternatively be improved by keeping the noise unchanged and adding a deterministic excitation selected in accordance with Melnikov theory, rather than by increasing the noise. We present qualitative arguments and results of numerical simulations according to which the extension we propose (a) significantly improves our ability to enhance SNR, (b) broadens the range of phenomena explainable by SR, and (c) allows the development of effective practical devices for enhancing SNR. Also, since Melnikov theory provides information on excitation frequencies that are effective in increasing a system's characteristic rate, a chaotic dynamics approach makes it possible to assess the role of the excitation's spectral density in the enhancement of the SNR, a problem of current interest in classical SR for which other available approaches can be unwieldy [9].

Section II describes the class of systems for which our approach is applicable and reviews briefly pertinent material on Melnikov theory. Section III considers the case of a bistable deterministic system excited by a sum of two harmonic terms. Chaotic behavior in this system is associated with a broadband spectrum on the basis of which the output

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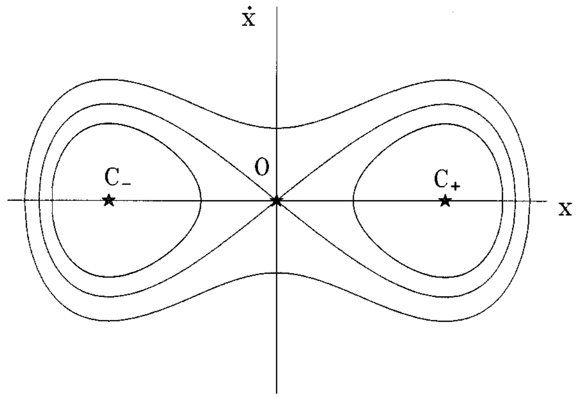

FIG. 1. Phase plane diagram for unperturbed system.

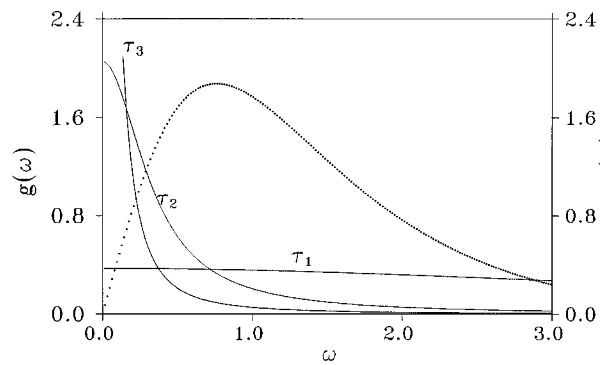

FIG. 2. Melnikov scale factor $S_M(\omega)$ for double-well potential (dotted line) and normalized power spectra $g(\omega)$ of stochastic excitation $\mathcal{R}(t)$ for three different correlation times τ (solid lines): $\tau_1=0.2$, $\tau_2=3.0$, $\tau_3=12.0$.

SNR can be defined, and we show how Melnikov theory can be used to enhance the SNR in this case. Section IV is devoted to classical SR and shows how Melnikov theory can be used to assess the effect of the spectral density of the noise on SNR enhancement. Section V shows that, for a system with signal and noise, the output SNR can be increased more effectively by adding to the system a harmonic excitation selected in accordance with Melnikov theory, rather than by increasing the noise. Section VI shows how the method described in Sec. V can be used to develop a nonlinear transducing device for enhancing SNR. Section VII presents our conclusions.

II. DYNAMICAL SYSTEMS AND NECESSARY CONDITION FOR CHAOS

We consider second-order dynamical systems described by the equation

$$\ddot{x}(t)=-\beta\dot{x}(t)-V'(x)+G(t), \tag{1}$$

where $V(x)$ is a potential function. The unperturbed counterpart of Eq. (1) is the Hamiltonian system

$$\ddot{x}=-V'(x). \tag{2}$$

We assume that Eq. (2) has a hyperbolic fixed point [6] connected to itself by a homoclinic orbit or two hyperbolic fixed points connected by a heteroclinic orbit. As an example, we consider in this paper the Duffing-Holmes equation, which has a double-well potential

$$V(x)=-1/2x^2+1/4x^4. \tag{3}$$

Equation (2) with the potential (3) has the homoclinic orbits shown in Fig. 1. The homoclinic orbits constitute a separatrix, that is, a curve separating motions that evolve around the centers C_- or C_+ and can never cross the potential barrier from motions that evolve around the hyperbolic fixed point O and cross the potential barrier periodically (Fig. 1). For the potential (3) integration of Eq. (2) with initial conditions $x=0$, $\dot{x}=0$ yields the following expressions for the homoclinic orbits:

$$x_0(t)=\pm\sqrt{2}\,\mathrm{sech}(t), \quad \dot{x}_0(t)=\pm\sqrt{2}\,\mathrm{sech}(t)\tanh(t). \tag{4}$$

We now review briefly basic Melnikov theory results pertaining to systems with periodic, quasiperiodic, and stochastic excitation. Let us assume first that the excitation is periodic, that is, in Eq. (1) $G(t)\equiv A_0\sin(\omega_0 t)$. The Smale-Birkhoff theorem states that the necesssary condition for the occurrence of chaos is that the Melnikov function induced by the perturbation have simple zeros. For the Duffing system this condition is the Melnikov inequality

$$-4/3\beta+A_0S_M(\omega_0)>0, \tag{5}$$

where

$$S_M(\omega)=\sqrt{2}\pi\omega\,\mathrm{sech}(\pi\omega/2) \tag{6}$$

is a system property known as the Melnikov scale factor [7]. For the Duffing oscillator $S_M(\omega)$ is shown in Fig. 2. [Also included in Fig. 2 are plots $g(\omega)$ to be defined later.] Next we assume that the excitation consists of the quasiperiodic sum

$$G(t)\equiv A_0\sin(\omega_0 t+\phi_0)+A_a\sin(\omega_a t)+\sum_{k=1}^{K}a_k\sin(\omega_k t+\varphi_k). \tag{7}$$

For this case a generalization of the Smale-Birkhoff theorem [7] yields as the necessary condition for chaos the Melnikov inequality

$$-4\beta/3+A_0S_M(\omega_0)+A_aS_M(\omega_a)+\sum_{k=1}^{K}a_kS_M(\omega_k)>0. \tag{8}$$

Finally, we assume that the system's excitation is

$$G(t)\equiv A_0\sin(\omega_0 t+\phi_0)+A_a\sin(\omega_a t)+\sqrt{2D\beta}\mathcal{R}(t), \tag{9}$$

where $\mathcal{R}(t)$ is a Gaussian process with unit variance and spectral density $g(\omega)$. Over any finite time interval, however large, each realization of the process $\mathcal{R}(t)$ may be approximated as closely as desired [10] by a sum

$$\mathcal{R}_N(t)=\sum_{k=1}^{K}b_k\sin(\omega_k t+\varphi_k), \tag{10}$$

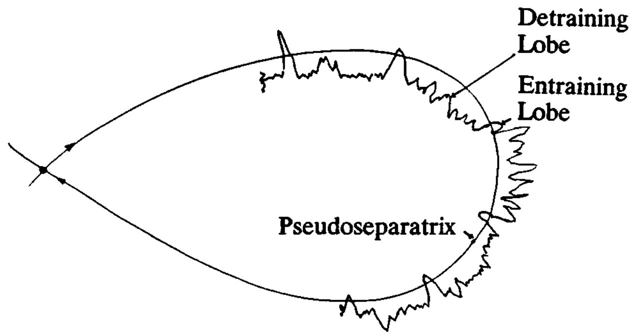

FIG. 3. Part of phase plane diagram showing intersecting stable and unstable manifolds of stochastically excited system.

so that the Melnikov inequality, that is, the necessary condition for chaos, can be written as Eq. (8), where $a_k = \sqrt{2D\beta} b_k$. In Eq. (10), $b_k = \sqrt{g(\omega_k)\Delta\omega}$, φ_k are randomly chosen phases of uniform distribution on the interval $[0,2\pi]$ and $\omega_k = k\Delta\omega$; $\Delta\omega = \omega_{\max}/K$; and $\omega_{\max}$ is the frequency beyond which the spectrum vanishes (the cutoff frequency). Note also that in the Melnikov inequality contributions of terms with high frequencies are suppressed owing to the exponential decay of $S_M(\omega)$ with increasing ω.

For the damped, forced system, the existence in a plane of section of a transverse point of intersection between the stable and unstable manifolds implies the existence of an infinity of intersection points. Areas bounded by segments of stable and unstable manifolds between two successive intersection points are termed lobes. A set of lobe segments forming a shape roughly similar to the shape of the unperturbed orbit's homoclinic orbit is termed a pseudoseparatrix [7]. Unlike the homoclinic orbit (i.e., unlike the separatrix of Fig. 1), the pseudoseparatrix is permeable, that is, it can allow motions occurring within a well to exit from that well. The transport of phase space across the pseudoseparatrix is affected by detraining and entraining lobes, and is referred to as chaotic transport [7]. [Detraining (entraining) lobes are lobes that will cross or have crossed into the exterior (interior) region bounded by the pseudoseparatrix [6] — see Fig. 3]. The strength of the chaotic transport, and therefore the characterisitc rate α, increases as the left-hand side of Eq. (8) becomes larger [7]. This is true regardless of whether the excitation is deterministic or stochastic. Moreover, again regardless of whether the excitation is deterministic or stochastic, a qualitative feature of the chaotic motions featuring escapes is that their spectral densities have a broadband portion with significant energy content at and near the system's characteristic rate α. This follows from the topological conjugacy of the deterministically or stochastically induced chaotic motions to a shift map, which is characterized, among other properties, by the existence of nonperiodic orbits.

In the following sections we use the qualitative considerations summarized above to examine SNR enhancement for the following types of excitation: deterministic excitation consisting of a harmonic signal and an added harmonic, excitation consisting of a harmonic signal and noise, and excitation consisting of a harmonic signal, an added harmonic, and noise.

III. SNR ENHANCEMENT FOR A BISTABLE DETERMINISTIC SYSTEM

Let us assume that the excitation is a sum of a harmonic signal and an added harmonic, that is, in Eq. (1) $G(t) \equiv A_0 \sin(\omega_0 t) + A_a \sin(\omega_a t)$. The system is therefore deterministic with, in general, quasiperiodic excitation. The necessary condition for chaos is given by Eq. (8) in which $a_1 = a_2 = \cdots = a_K = 0$. We choose A_0 so that, for $A_a = 0$, the motion is confined to one well. In accordance with Melnikov theory this will be the case if the Melnikov inequality given by Eq. (5) is not satisfied. We now add the excitation $A_a \sin(\omega_a t)$. For a certain region R_a of the parameter space $[A_a, \omega_a]$, the system can experience chaotic motion with jumps over the potential barrier. The Melnikov scale factor $S_M(\omega)$ provides the information needed to select frequencies ω_a such that the added excitation is effective in inducing chaotic behavior. It follows from Eqs. (6) and (8) that ω_a should be equal or close to the frequency for which $S_M(\omega)$ is largest; see Fig. 2. For chaotic motions the spectral density has (i) peaks at the fundamental excitation frequencies ω_0 and ω_a and linear combinations thereof, and (ii) a broadband portion due to the chaotic nature of the response.

Given the existence in the spectrum of a broadband portion qualitatively similar to that present in the case of classical SR, it is reasonable to expect that the synchronization-like phenomenon that occurs in the classical SR case would similarly occur for the deterministically excited chaotic system. This was verified by numerical simulation for a large number of cases. As a typical example, we consider the case $\beta = 0.316$, $A_0 = 0.095$, $\omega_0 = 0.0632$ [for these values Eq. (5) is not satisfied], and $\omega_a = 1.1$. Spectral densities of motions with these parameters and $A_a = 0.263$, 0.287, and 0.332, are shown in Figs. 4(a), 4(b), and 4(c), respectively. (Note that owing to the broadband portion of the spectrum a SNR can be defined for the output just as in the case of classical SR.) For Fig. 4(b), $\alpha = 0.0672$ is close to the signal frequency $\omega_0 = 0.0632$. The energy in the broadband portion of the spectrum is depleted, while the energy at the signal's frequency is enhanced, with respect to their respective counterparts in Figs. 4(a) and 4(c), for which $\alpha = 0.0395$ and 0.158, respectively. The synchronizationlike phenomenon noted for classical SR is thus clearly evident in Fig. 4(b). We also verified that the motions of Figs. 4(a), 4(b), and 4(c) are indeed chaotic (i.e., their largest Lyapounov exponents, estimated as in [11], are positive). Figure 5 shows the dependence of the SNR on A_a. Note that the plot of Fig. 5 is similar qualitatively to plots of the SNR versus noise intensity D for classical SR.

IV. NOISE SPECTRUM EFFECT ON SNR FOR CLASSICAL SR

We now consider a system excited by noise and a harmonic signal, which, by itself, cannot induce jumps. To assess the effect of the shape of the noise spectrum on the SNR we use the fact that the Melnikov scale factor $S_M(\omega)$ is a measure of the degree to which a harmonic excitation or a frequency component can be effective in inducing chaotic behavior.

On the one hand the noise excitation increase has an un-

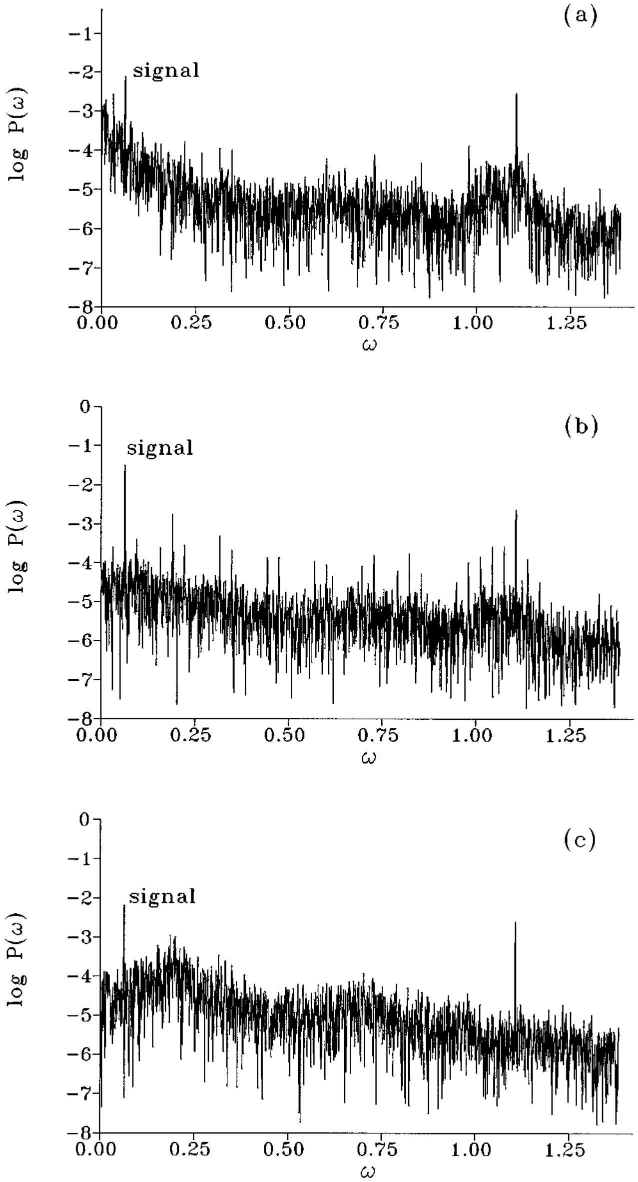

FIG. 4. Power spectra of system with no stochastic excitation ($D=0$). The amplitude A_0 and frequency ω_0 of the signal are kept constant. The system is subjected to an additional harmonic excitation with frequency $\omega_a=1.1$ and amplitude A_a: (a) $A_a=0.263$, (b) $A_a=0.287$, (c) $A_a=0.332$ (logarithms in base 10).

favorable effect on the SNR insofar as it increases the output noise level. It is this unfavorable effect that renders classical SR an apparent paradox. On the other hand, the noise excitation has a favorable effect, that is, it brings the rate α in line with the frequency ω_0 and thus allows the occurrence of the synchronizationlike phenomenon, which more than makes up for the increase of the noise. It is reasonable to expect that the smaller the power of the noise that helps to bring about a rate $\alpha \approx \omega_0$, the better the SNR will be.

We recall that the larger the left-hand side of Eq. (8), the stronger is the chaotic transport across the pseudoseparatrix, and therefore the larger is the rate α [7,8]. Recall that in Eq. (8) (in which it is now assumed $A_a=0$), $a_k=\sqrt{2D\beta}\sqrt{g(\omega_k)\Delta\omega}$. It is therefore clear from Eq. (8) that for any given power of the stochastic excitation $2D\beta$, the

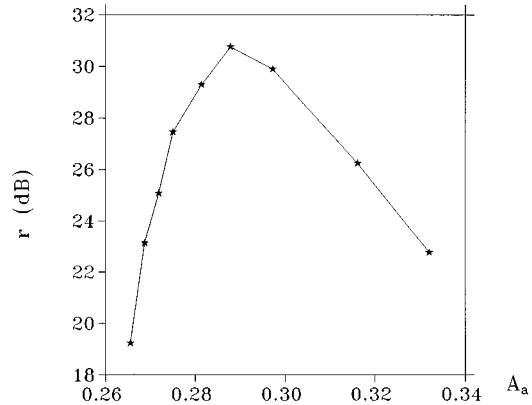

FIG. 5. Signal-to-noise ratio [1] r vs amplitude of added harmonic excitation A_a, without stochastic component ($D=0$).

left-hand side of Eq. (8) becomes larger and the rate α increases, as the integral

$$I=\int_0^{\omega_{\max}} g(\omega)S_M^2(\omega)d\omega \qquad (11)$$

increases. [The integrand in Eq. (11) is the ordinate of the contribution of the stochastic excitation to the spectrum of the Melnikov process [8].] We thus obtain the interesting qualitative result that, for a given Melnikov scale factor $S_M(\omega)$ and a given power of the stochastic excitation, the rate α increases as the spectral power of the excitation is distributed nearer to the frequency of $S_M(\omega)$'s peak, $\omega_{\rm pk}$ (the greatest effectiveness being achieved by a single component with frequency equal or close to $\omega_{\rm pk}$).

We now illustrate the usefulness of this result for a system with classical SR [i.e., one for which in Eq. (9) $A_a=0$, $D>0$]. We assume $\mathcal{R}(t)$ has the Lorentzian spectral distribution $g(\omega)=\gamma/\tau(1+\omega^2\tau^2)^{-1}$ cut off at the frequency $\omega_{\max}$; τ is the correlation time and γ is a normalization constant such that the variance of $\mathcal{R}(t)$ is unity. Figure 2 shows spectra $g(\omega)$ for three values of τ and $\omega_{\max}=3.0$. As can be seen in Fig. 2, the Melnikov scale factor $S_M(\omega)$ would in practice suppress contributions of components with frequencies $\omega>\omega_{\max}$. Therefore our use of a cutoff point, which is motivated merely by computational convenience, does not affect the significance of our results. We are interested in the effect on the peak SNR of the parameter τ (i.e., of the shape of the noise spectrum).

We examine first the case $\tau=\tau_1=0.2$. Examples of averaged output spectra $P(\omega)$ for $A_0=0.3$, $\omega_0=0.069$, $\omega_{\max}=3.0$, $\beta=0.25$ are shown in Figs. 6(a)–6(c) for power $D=0.01$, 0.04, and 0.22, respectively. The averaging was performed over 225 noise realizations approximated by Eq. (10) with $100<K<500$. Note that $A_0<4\beta/3S_M(\omega_0)$, so that no chaotic behavior can be induced by the periodic signal alone. However, it was verified that, for the noise realizations used to obtain the results of Figs. 6(a)–6(c), the Melnikov inequality given by Eq. (8) was satisfied, and that the respective motions were chaotic. Energy transfer to the signal frequency was found to be highest when the rate α for the

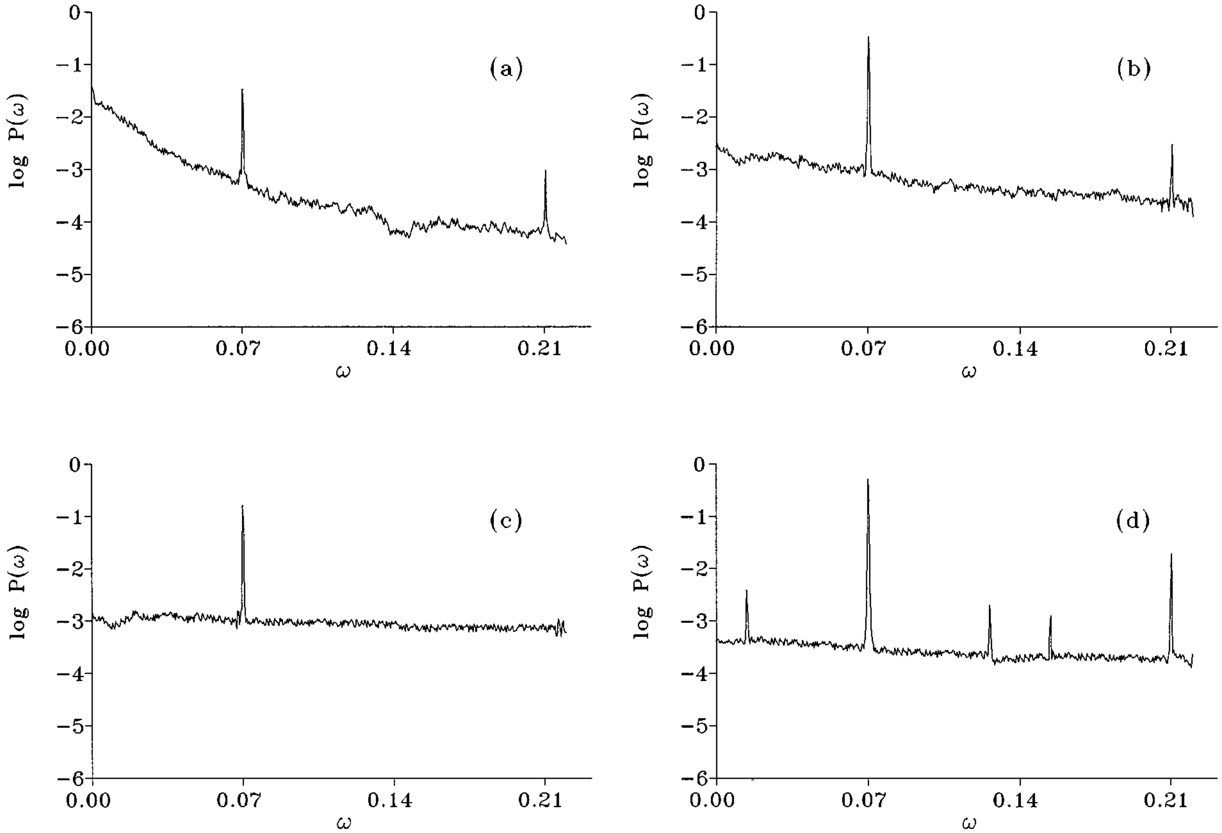

FIG. 6. Averaged power spectra of output for stochastically excited system: (a)–(c) Increasing noise intensity D and $A_a=0$. (d) The same noise intensity D as in (a), and $A_a=0.23$. Noise correlation time $\tau=0.2$ in all cases (logarithms in base 10).

chaotic motion was close to the signal frequency; see Fig. 6(b). The dependence of SNR on noise intensity is plotted in Fig. 7.

Figure 7 also shows similar plots for $\tau=3$ and 12, the parameters A_0, ω_0, $\omega_{\max}$, and β being the same as for the case $\tau=0.2$. We note that for $\tau=0.2$, 3, and 12, $I=0.626$, 0.411, and 0.157, respectively. As expected, the peak SNR is smaller and occurs at higher values of D for larger correlation times τ, that is for spectral shapes with energy content distributed farther from the frequency ω_{pk} (see Fig. 2) or, equivalently, for smaller values of I. We note that similar effects were observed experimentally; see [12].

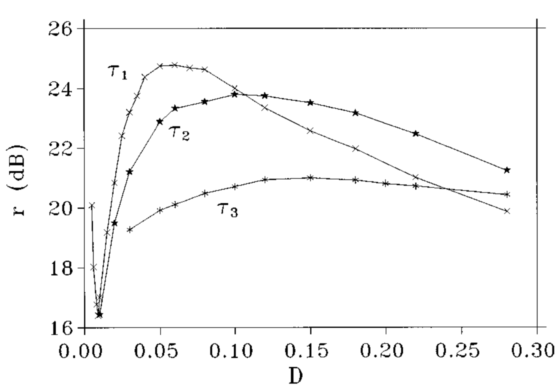

FIG. 7. Signal-to-noise ratio [1] r vs noise intensity D for the three noise correlation times τ of Fig. 2.

V. SYSTEM WITH HARMONIC SIGNAL AND NOISE: SNR ENHANCEMENT BY ADDING HARMONIC EXCITATION

The results of the preceding sections suggest the following method for improving SNR. Assume that $A_a=0$, and that for a set of values A_0, ω_0, β, and D the system has low SNR. We could improve the SNR by increasing D, as illustrated earlier. However, it is more effective to increase the SNR by keeping D unchanged and adding an excitation $A_a\sin(\omega_a t)$ such that (1) ω_a is equal or close to the frequency of $S_M(\omega)$'s peak and (2) A_a is so chosen as to bring about a characteristic rate comparable to the signal frequency. An example is shown in Fig. 6(d), for which all parameters and the normalized spectrum $g(\omega)$ are the same as for Fig. 6(a), except that the system is subjected to an added excitation with amplitude $A_a=0.23$ and frequency $\omega_a=1.1$. This approach to increasing SNR is seen to be quite effective. Note that the added harmonic excitation induces subharmonics and superharmonics that are well separated from the signal and can therefore be filtered out by a suitable passband filter.

VI. PROPOSED NONLINEAR TRANSDUCING DEVICE FOR ENHANCING SNR

We now describe the principle of a nonlinear transducing device for improving a signal's SNR based on the method

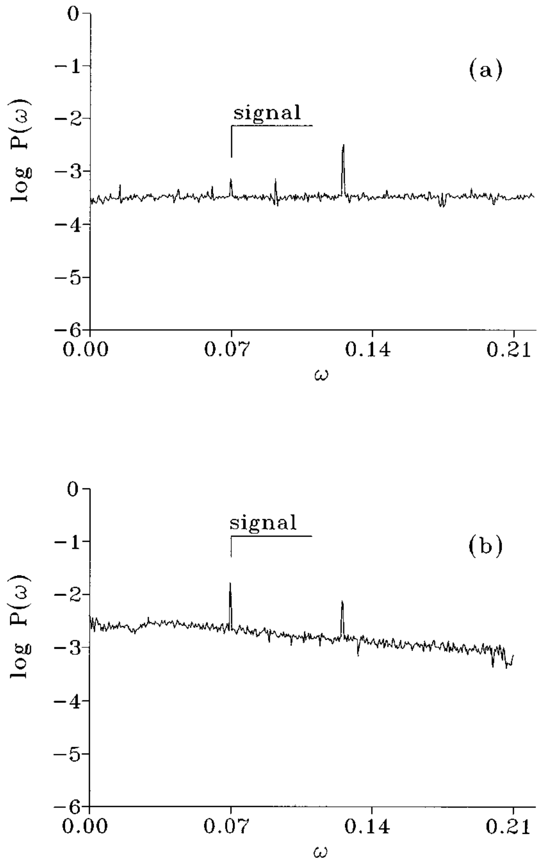

FIG. 8. Averaged power spectra of (a) input, consisting of stochastic excitation $\mathcal{R}(t)$, harmonic signal with frequency $\omega_0 = 0.069$, and additional harmonic excitation with frequency $\omega_a = 1.1$. (b) Output of transducing device; see text for details (logarithms in base 10).

discussed in the preceding section. Consider a signal for which the SNR is unsatisfactory. The signal and the attendant noise — from which we filter out components well separated from the signal, that is, components with frequencies exceeding, say, three times the signal frequency — are used to excite the transducing device, consisting, for example, of a Duffing oscillator. The SNR of the output will in general be poor, but under certain conditions it can be improved by the addition of a harmonic excitation with frequency equal or close to the frequency of the Melnikov scale factor's peak. The role of the added harmonic excitation is to bring about a chaotic motion with characteristic rate close to the signal frequency. To illustrate the principle of the device, we show in Fig. 8(a) the spectrum of a signal $A_0 \sin \omega_0 t$, $A_0 = 0.05$, $\omega_0 = 0.069$ in the presence of noise $\sqrt{2D\beta}\mathcal{R}(t)$, with $\tau = 0.2$ (see Fig. 2), $\beta = 0.25$, and $D = 0.72$. Using a low passband filter, we filter out the noise components with frequencies larger than three times the frequency of the signal. The signal and the noise left after the filtering [i.e., the noise $\sqrt{2D\beta}\mathcal{R}(t)H(3\omega_0)$, where H denotes the Heaviside step function] are used as input to a nonlinear system described by Eq. (1). For $A_a = 0$ the SNR of the output is not better than the input SNR. However, by subjecting the nonlinear system to the additional excitation $A_a \sin \omega_a t$ ($A_a = 0.23$ and $\omega_a = 1.1$) we obtain the result shown in Fig. 8(b).

VII. CONCLUSIONS

The chaotic dynamics approach adopted in this paper provides a unifying framework wherein classical stochastic resonance — the enhancement of the SNR achieved by increasing the noise intensity — is viewed as a particular case of a type of chaotic behavior that includes, as another particular case, the enhancement of the SNR by adding a harmonic excitation while leaving the noise unchanged. By making it possible to apply Melnikov theory, the chaotic dynamics approach allows the use of qualitative results related to the fundamental fact that for each of those particular cases the system motion is topologically conjugate to a shift map. One of these qualitative results is the existence, independent of the deterministic, stochastic, or mixed character of the excitation, of a broadband portion of the output spectrum, which allows the occurrence of a synchronizationlike phenomenon that is the key to the enhancement of the SNR. Another qualitative result is that the effectiveness of a harmonic excitation or frequency component in promoting chaotic motion with jumps over a potential barrier depends on the system's Melnikov scale factor.

These qualitative results suggested the investigation of the alternative mechanism for enhancing SNR, wherein the noise intensity is left unchanged and a harmonic excitation is added instead. This mechanism is more effective — allows a better SNR to be obtained — than the mechanism that relies on increasing the noise intensity. The alternative mechanism we investigated allows the development of a practical device that accepts a signal with low SNR and converts it into an output with significantly greater SNR. Our alternative mechanism may also explain some natural phenomena more plausibly than is the case for classical stochastic resonance. For example, experiments on crayfish mechanoreceptors have shown that the capability of the latter to detect weak signals in a noisy environment could be explained by classical stochastic resonance [13]. However, one might argue that (a) classical stochastic resonance is relatively inefficient, and (b) a neuron is unlikely to control the level of external noise and increase it for the purpose of SNR enhancement. It is therefore reasonable to also consider the possibility that the neuron's capability to increase the SNR is due to the action of periodic or nearly periodic physiological cycles. Our alternative mechanism for enhancing SNR could be relevant in this context.

Qualitative results of Melnikov theory also suggested a transparent and convenient method for assessing the effect of the spectral density of the noise on SNR enhancement by classical SR. From that method it follows that the closer the spectral power of the noise is distributed to the Melnikov scale factor's peak, the more effective the associated spectral shape is in enhancing the SNR. Numerical simulations, of which typical examples are included in the paper, support the qualitative results we just summarized.

ACKNOWLEDGMENTS

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[1] The signal to noise ratio (SNR) is denoted by r and is expressed in dB as $r=10\log_{10}(S/N)$, where S and N are, respectively, the ordinate of the output spectrum and the ordinate of the broadband output spectrum at the signal frequency ω_0.

[2] For a review, see F. Moss, in *Some Problems in Statistical Physics*, edited by G.H. Weiss, Frontiers in Applied Mathematics (SIAM, Philadelphia, 1992).

[3] M.I. Dykman, D.G. Luchinsky, R. Mannella, P.V.E. McClintock, N.D. Stein, and N.G. Stocks, J. Stat. Phys. **70**, 463 (1993); **70**, 479 (1993).

[4] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).

[5] B. Shulgin, A. Neiman, and V. Anishchenko, Phys. Rev. Lett. **75**, 4157 (1996).

[6] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1990).

[7] D. Beigie, A. Leonard, and S. Wiggins, Nonlinearity **4,** 775 (1991); S. Wiggins, *Chaotic Transport in Dynamical Systems* (Springer-Verlag, New York, 1992).

[8] M. Frey and E. Simiu, Physica D **63**, 321 (1993); S.R. Hsieh, A.W. Troesch, and S.W. Shaw, Proc. R. Soc. London A **446**, 195 (1994); E. Simiu and M. Frey, J. Eng. Mech. **122**, 263 (1996); M. Franaszek and E. Simiu, Phys. Lett. A **205**, 137 (1995); Y.R. Sivathanu, C. Hagwood, and E. Simiu, Phys. Rev. E **52**, 4669 (1995); M. Frey and E. Simiu (unpublished); E. Simiu, J. Appl. Mech. **63**, 429 (1996).

[9] P. Hänggi, P. Jung, C. Zerbe, and F. Moss, J. Stat. Phys. **70**, 25 (1993).

[10] S.O. Rice, in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954).

[11] P. Bergé, Y. Pomeau, and C. Vidal, *Order Within Chaos* (Wiley, New York, 1984), p. 129.

[12] L. Gammaitoni, E. Menichella-Saetta, S. Santucci, F. Marchesoni, and C. Presilla, Phys. Rev. A **40**, 2114 (1989).

[13] J.K. Douglass, L. Wilkens, E. Pantazelou, and F. Moss, Nature **365**, 337 (1993); K. Wiesenfeld and F. Moss, *ibid.* **373**, 33 (1995).